Lianne de Jonge

Exam Calculus of Variations and Optimal Control 2017-18

Date	:	23-01-2017
Place	:	Martini Plaza
Time	:	09.00-12.00

The exam is OPEN BOOK; you can use all your books/papers/notes; but NO internet connection.
You are supposed to provide arguments to all your answers, and to explicitly refer to theorems/propositions whenever you use them.

1. Consider minimization of

$$\int_0^1 \left(\dot{x}^2(t) - x(t)\right) dt, \quad x(0) = 0, x(1) = 1,$$

over all functions $x : [0,1] \to \mathbb{R}$.

(a) Determine the Euler-Lagrange equation for this problem.

(b) Solve the Euler-Lagrange equation.

(c) Show that the obtained solution is optimal.

2. Consider the system

$$\dot{x} = x(1-u), \quad x(0) = 1,\ x(1) = 4e$$

with cost criterion

$$\int_0^1 -\ln(x(t)u(t))\, dt$$

(a) Determine the Hamiltonian, and the Hamiltonian equations of the Minimum principle; including boundary conditions.

(b) Show that

$$u(t) = -\frac{1}{p(t)x(t)},$$

with p the co-state, is the candidate optimal control.

(c) Substitute this u into the Hamiltonian equations, and solve for the optimal functions $p^*(t)$ and $x^*(t)$, and eventually $u^*(t)$.

3. Consider the scalar system

$$\dot{x} = ax + u,\ x(0) = x_0, \quad x, u \in \mathbb{R}, a > 0,$$

with a quadratic cost criterion

$$\int_0^\infty \left(x^2(t) + ru^2(t)\right) dt,\ r > 0$$

(a) Write down the algebraic Riccati equation. Compute the optimal control in feedback form, as well as the minimal cost as a function of the initial condition x_0.

(b) Let $r \to 0$ (the so-called 'cheap control' problem). Prove that the eigenvalue of the closed-loop system moves off to $-\infty$. What is the interpretation of this?

(c) Let $r \to \infty$ (the 'expensive control' problem). Prove that in this case the eigenvalue of the closed-loop system tends to $-a$. Consider the corresponding Hamiltonian matrix and interpret the result from this point of view.

(d) Finally, assume instead $a < 0$, and let again $r \to \infty$. What is happening in this case, and what is now the interpretation?

4. Consider a mass-spring system with so-called 'hardening' spring. The system is given by

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -k_1 q - k_2 q^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad k_1 > 0, k_2 > 0,$$

where u is an external force.

(a) Determine the equilibrium point(s) of the system for $u = 0$.

(b) What can be said about the stability of the origin for $u = 0$ using Lyapunovs first method (linearization)? Motivate your answer!

(c) The total energy of the mass-spring system is given by

$$E(q, \dot{q}) = \frac{1}{2}\dot{q}^2 + \frac{1}{2}k_1 q^2 + \frac{1}{4}k_2 q^4$$

What can be said about the stability of the origin for $u = 0$ using Lyapunov's direct (or second) method? Motivate your answer!

(d) Prove that the asymptotic stabilization approach using a Lyapunov function leads to the feedback $u = -\dot{q}$.
Prove by LaSalle's invariance principle that the origin of the controlled system resulting from $u = -\dot{q}$ is indeed asymptotically stable. What can be said about *global* asymptotic stability?

Distribution of points: Total 100; Free 10.

1. a: 5, b: 8, c: 7.
2. a: 8, b: 4, c: 8.
3. a: 7, b: 6, c: 7, d: 5.
4. a: 4, b: 6, c: 7, d: 8.